Total marks 36
Attempt questions 1-3
Answer each question in a SEPARATE writing booklet.

Question 1 (12 Marks) Use a SEPARATE writing booklet **Marks**

(a) If the velocity of a particle is given by $v = \frac{1}{2t+1} ms^{-1}$ and it is initially at $x = 1$, find an expression for the displacement x in terms of t. **2**

(b) A particle undergoes Simple Harmonic Motion about the origin O. Its displacement, *x* centimetres from the origin at time *t* seconds, is given by $x = 4\cos\left(2t + \frac{\pi}{3}\right)$

(i) Express the acceleration as a function of displacement. **1**

(ii) Write down the amplitude of the motion. **1**

(iii) Find the maximum speed and the time at which it first occurs. **2**

(c) The acceleration of an object moving along the *x*-axis is given by $a = 10x - 5x^3\ cm/s^2$. It is released from rest at $x = 2$.

(i) If v is the velocity, show that $v^2 = x^2\left(10 - 2.5x^2\right)$. **3**

(ii) In which direction does the object first move? Show why. **2**

(iii) Where does the object next come to rest? **1**

Question 2 12 Marks Use a SEPARATE writing booklet Marks

(a) A baseball pitcher throws his fastball ball at $40\ metres/\sec$. He releases the ball 1.8 metres above the ground at an angle of 2° above the horizontal and at a distance of 18.4 metres from the batter.

The equations of motion for the ball are: $\ddot{x} = 0$ and $\ddot{y} = -10$.

Take the origin to be the ground directly below the point where the pitcher releases the ball.

(i) Using calculus, prove that the coordinates of the ball at time t are given by: **3**

$$x = 40t\cos\left(2^\circ\right) \text{ and } y = -5t^2 + 40t\sin\left(2^\circ\right) + 1.8$$

(ii) Find the time it takes for the ball to reach the batter. **1**

(iii) To be a "strike" for a particular batter, the ball must pass the batter between 53cm and 115cm above the ground. Was this ball a strike? Show your reasoning clearly. **2**

(iv) Find the angle (to the nearest minute) at which the ball arrives at the batter. **2**

(b) Given the equations of motion of an cannonball fired from the origin with initial speed $V\ m/s$ at an angle of α° are given by:

$$x = Vt\cos\alpha \text{ and } y = -\frac{1}{2}gt^2 + Vt\sin\alpha,$$

(i) Show that the Cartesian equation of the path of the object are given by $y = -\dfrac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$ **2**

(ii) Show that the range on level ground is given by $\dfrac{V^2}{g}\sin 2\alpha$. **1**

(iii) If $V = 33\ m/s$ and taking $g = 10\ m/s^2$, find the maximum distance that the cannonball can be fired. **1**

Question 3 (12 Marks) Use a SEPARATE writing booklet **Marks**

(a) Find the coefficient of x^6 in the expansion of $\left(1-2x^2\right)^5$ **2**

(b) Using the fact that $(1+x)^4(1+x)^9=(1+x)^{13}$, show that **2**

$${}^4C_0\,{}^9C_4+{}^4C_1\,{}^9C_3+{}^4C_2\,{}^9C_2+{}^4C_3\,{}^9C_1+{}^4C_4\,{}^9C_0={}^{13}C_4$$

(c) Let $(2+5x)^{20}=\sum_{k=0}^{20}T_k$ where T_k is the term in x^k.

(i) Show that $\dfrac{T_{k+1}}{T_k}=\dfrac{100-5k}{2k+2}x$ **2**

(ii) If $x=\dfrac{1}{3}$, find the greatest term in factored form. **3**

(d) (i) Use the binomial theorem to obtain an expansion for **1**

$(1+x)^{2n}+(1-x)^{2n}$ where n is a positive integer.

(ii) Hence evaluate $1+{}^{20}C_2+{}^{20}C_4+\ldots\ldots+{}^{20}C_{20}$. **2**

End of Paper

Question 1.

(a) $v = \dfrac{1}{2t+1}$ $t=0,\ x=1$.

$x = \frac{1}{2}\int \frac{2}{2t+1}\,dt$

$x = \frac{1}{2}\ln(2t+1) + c$ ✓

when $t=0$ $x=1$

$1 = \frac{1}{2}\ln 1 + c$

$c = 1$

$\therefore x = \frac{1}{2}\ln(2t+1) + 1$ ✓

(b) $x = 4\cos(2t + \pi/3)$

$\dot{x} = -8\sin(2t + \pi/3)$

$\ddot{x} = -16\cos(2t + \pi/3)$

(i) $\ddot{x} = -4x$ ✓

(ii) amp is 4. ✓

(iii) max speed is where $\sin(2t + \pi/3) = 1$

and is 8 cm/sec ✓

which is when $2t + \pi/3 = \pi/2$

$2t = \pi/6$

$t = \pi/12$ secs ✓

(c) $a = 10x - 5x^3$ $t=0,\ v=0,\ x=2$

(i) $\dfrac{d\frac{1}{2}v^2}{dx} = 10x - 5x^3$ ✓

Integrating wrt x.

$\frac{1}{2}v^2 = 5x^2 - \frac{5}{4}x^4 + c$ ✓

when $v=0$ $x=2$.

$0 = 20 - 20 + c$ $\therefore c = 0$ ✓

$\therefore \frac{1}{2}v^2 = 5x^2 - \frac{5}{4}x^4$

$v^2 = 10x^2 - \frac{5}{2}x^4$

If just integrate wrt x not using $\dfrac{d\frac{1}{2}v^2}{dx}$ then no marks.

(ii) since initial velocity is zero, direction is determined by accelerat[ion]

at $t=0$ $a = 10\times2 - 5\times2^3$ ✓

$= -20$ cm/s^2

$\therefore$ ~~object~~ it will move to the left. ✓

Needs to show clearly why it moves to the left.

(iii) $v^2 = x^2(10 - \frac{5}{2}x^2)$

moving left from $x=2$, next stationary point is $x=0$ ✓

Q2

a) i) $\ddot{x} = 0$

$\dot{x} = \int dt$

$= c$

$t=0 \quad \dot{x} = 40\cos 2^\circ$

$x = \int 40\cos 2^\circ \, dt$

$= 40t\cos 2^\circ + c$

$t=0 \quad x=0$

$x = 40t\cos 2^\circ$ ✓

$\ddot{y} = -10$

$\dot{y} = \int -10\, dt$

$\dot{y} = -10t + c$

$t=0 \quad \dot{y} = 40\sin 2^\circ$

$\dot{y} = -10t + 40\sin 2^\circ$ ✓

$y = \int -10t + 40\sin 2^\circ \, dt$

$= -5t^2 + 40t\sin 2^\circ + c$

$t=0 \quad y = 1.8$

$y = -5t^2 + 40t\sin 2^\circ + 1.8$ ✓

ii) $x = 18.4$

$18.4 = 40t\cos 2^\circ$

$t = \frac{18.4}{40\cos 2^\circ}$ ✓

≈ 0.4602803904 sec

iii) $t = 0.4602803904$ ✓

$y = -5(0.46)^2 + 40(0.46)\sin 2^\circ + 1.8$ ✓

$= 1.38$ m

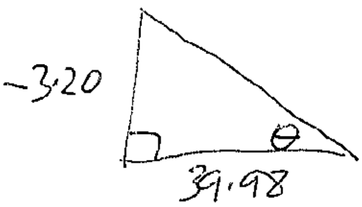
since it is greater than 115cm it is not a strike

iv) $t = 0.46$

$\dot{x} = 40\cos 2^\circ = 39.98$ ✓

$\dot{y} = -10(0.46) + 40\sin 2^\circ$

$= -3.20$



$\tan\theta = \frac{3.20}{39.98}$ ✓

$\theta \approx 4^\circ 35'$

Speed $= \sqrt{(-3.20)^2 + (39.98)^2}$

$\doteqdot 40.1$ m/s

b) i) $x = Vt\cos\alpha$ ①

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ ②

From ① $t = \frac{x}{V\cos\alpha}$ subin ② ✓

$y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha$ ✓

$= -\frac{1}{2}g\frac{x^2}{V^2}\sec^2\alpha + x\tan\alpha$

ii) Range $y=0$

~~...~~ $0 = -\frac{1}{2}g\frac{x^2}{V^2}\sec^2\alpha + x\tan\alpha$

$y=0$ or $\frac{-gx}{2V^2}\sec^2\alpha + \tan\alpha = 0$

$x = \frac{2V^2\tan\alpha}{g\sec^2\alpha}$

$= \frac{V^2}{g}2\cos^2\alpha\tan\alpha$ ✓

$= \frac{V^2}{g}2\sin\alpha\cos\alpha$

$= \frac{V^2}{g}\sin 2\alpha$

iii) Range $= \frac{33^2}{10}\times 1$ Max when $\alpha = \frac{\pi}{4}$

$= 108.9\text{m}$ ✓

Question 3

(a) $(1-2x^2)^5 = \sum_{i=0}^{5} \boxed{{}^5C_i(-2x^2)^i}$ ✓

$= \sum_{i=0}^{5} {}^5C_i(-2)^i x^{2i}$

For term in x^6 $2i=6$, ie $i=3$.

$\therefore$ coeff of $x^6 = {}^5C_3(-2)^3$

$= -80$. ✓

(b) $(1+x)^{2n}+(1-x)^{2n} = ({}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \dots + {}^{2n}C_{2n}x^{2n})$
$+ ({}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2 - \dots + {}^{2n}C_{2n}x^{2n})$

$(1+x)^{2n}+(1-x)^{2n} = 2({}^{2n}C_0 + {}^{2n}C_2x^2 + \dots + {}^{2n}C_{2n}x^{2n})$. ✓

sub $x=1$ in both side and $n=10$.

$(1+1)^{20} + 0 = 2({}^{2n}C_0 + {}^{2n}C_2 + \dots + {}^{2n}C_{2n})$. ✓

$2^{20} = 2(1 + {}^{20}C_2 + \dots + {}^{20}C_{20})$

$\therefore 1 + {}^{20}C_2 + {}^{20}C_4 + \dots + {}^{20}C_{20} = 2^{19}$ ✓

(c) LHS $= ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)({}^9C_0 + {}^9C_1x + {}^9C_2x^2 + {}^9C_3x^3 + {}^9C_4x^4 \dots)$ ✓

Note ${}^{13}C_4$ is coeff of x^4 in expansion of $(1+x)^{13}$ ✓

$\therefore$ find coeff of x^4 in expansion of $(1+x)^4(1+x)^9$.

ie ${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0$

Equating coefficients gives result.

(d)(i) In $(2+5x)^{20}$, $T_k = {}^{20}C_k\, 2^{20-k}(5x)^k$ ✓

$= {}^{20}C_k\, 2^{20-k}\, 5^k x^k$.

$T_{k+1} = {}^{20}C_{k+1}\, 2^{19-k}\, 5^{k+1} x^{k+1}$ ✓

$\frac{T_{k+1}}{T_k} = \frac{{}^{20}C_{k+1}\, 2^{19-k}\, 5^{k+1} x^{k+1}}{{}^{20}C_k\, 2^{20-k}\, 5^k x^k}$ ✓

$= \frac{5(20-k)}{2(k+1)} \times x$

$= \frac{(100-5k)}{2k+2} \times x$.

$\frac{{}^{20}C_{k+1}}{{}^{20}C_k} = \frac{20!}{(k+1)!(20-(k+1))!} \times \frac{k!(20-k)!}{20!}$

$= \frac{(20-k)}{(k+1)}$

(ii) If $x = \frac{1}{3}$ $\frac{T_{k+1}}{T_k} = \frac{100-5k}{6k+6}$

For greatest term find k such that

$\frac{T_{k+1}}{T_k} > 1$ ✓

ie $\frac{100-5k}{6k+6} > 1$

ie $100-5k > 6k+6$

$11k < 94$ ✓

$k < 8\frac{6}{11}$

ie $k=8$ is highest value for which this is true

$\therefore T_9 > T_8 > T_7$ etc. ie T_9 is greatest

$T_9 = {}^{20}C_9\, 2^{11} \times 5^9 \cdot \left(\frac{1}{3}\right)^9$ ✓